Efficient Eigen-Analysis of Large Delayed Cyber-Physical Power System

A Class of Semi-group Discretization-based Methods

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Outline

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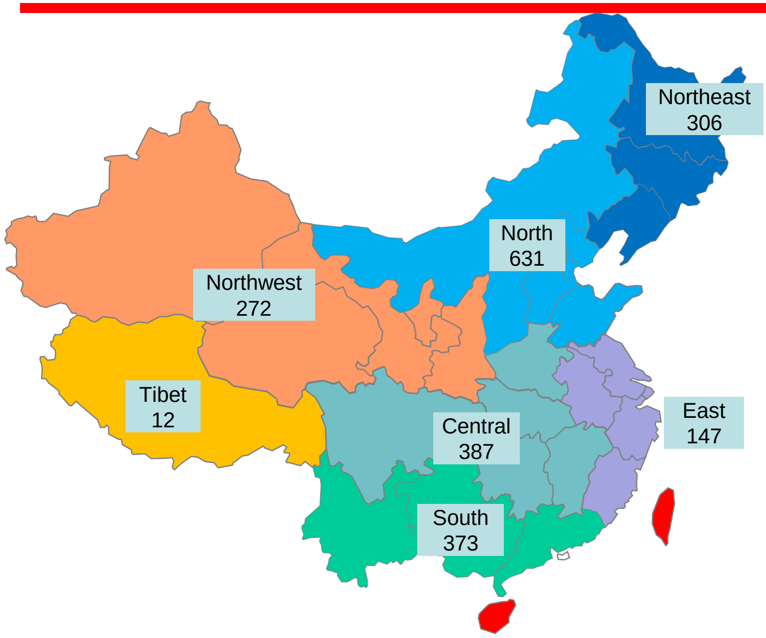
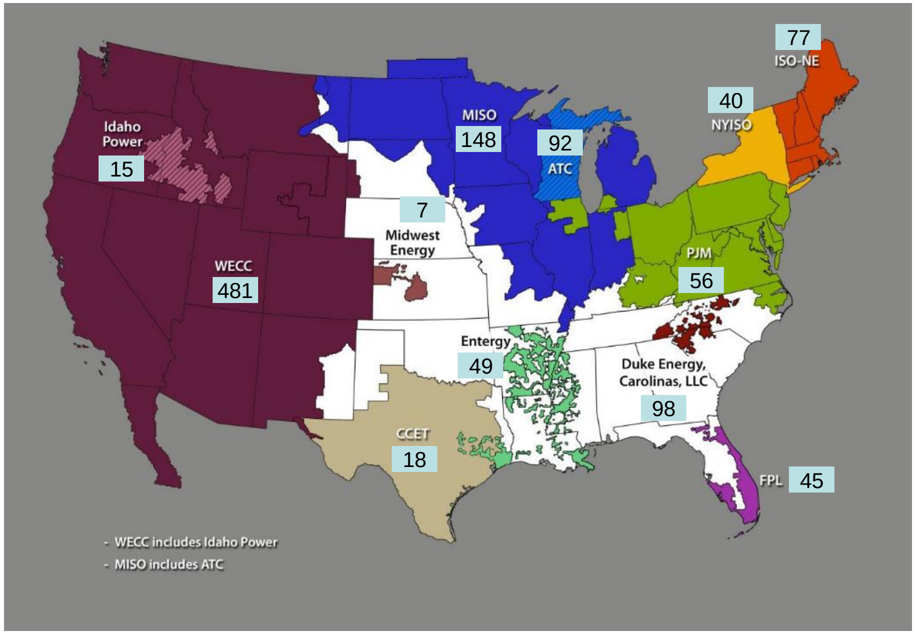
1. Background and Problem Formulation



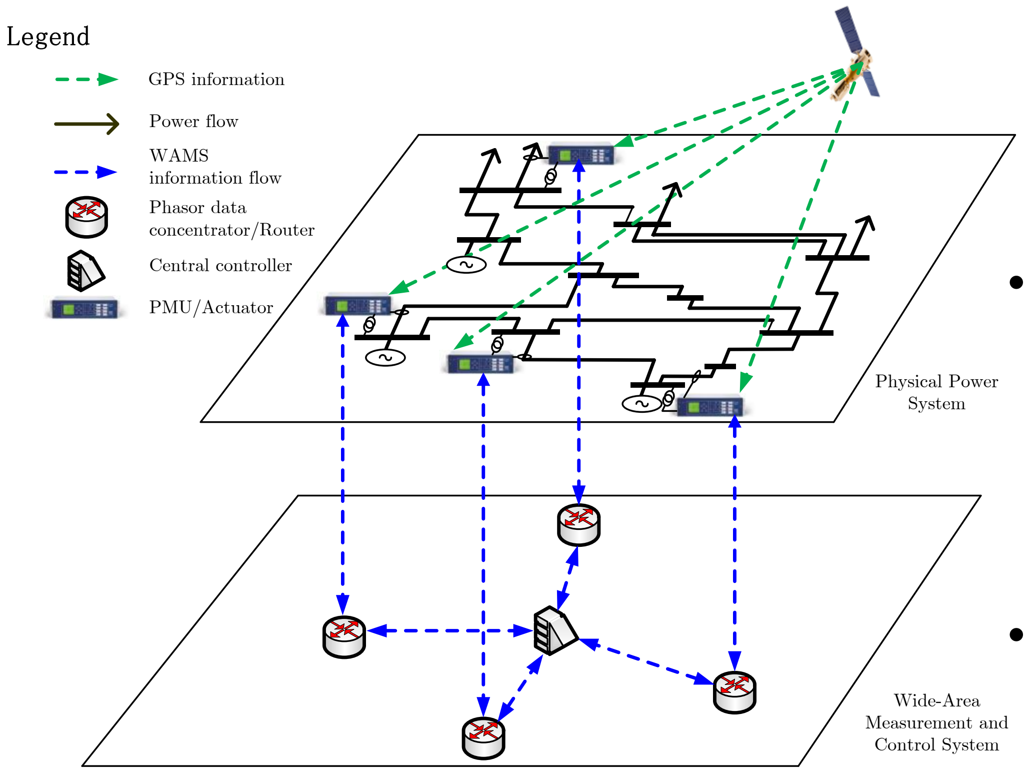
* As of 2012, over **2400** PMUs deployed in 500 kV and higher plants & substations of China.
* Source: Lu C, Shi B, Wu X, et al. Advancing China's smart grid: Phasor measurement units in a wide-area management system. *IEEE Power and Energy Magazine*, 2015, 13(5): 60-71.



* As of 03/31/2013, there were **1126** PMUs installed in US.
* Source: Department of Energy. Synchrophasor Technologies and their Deployment in Recovery Act Smart Grid Programs, August 2013.

- Thousands of phasor measurement units (PMUs) have been deployed in the transmission level all over the world, which provide a new measure for measuring, monitoring and control of the physical power system.
- Especially, wide-area damping controllers (WADCs) can effectively stabilize both local and interarea low frequency oscillations.

Background(cont'd)

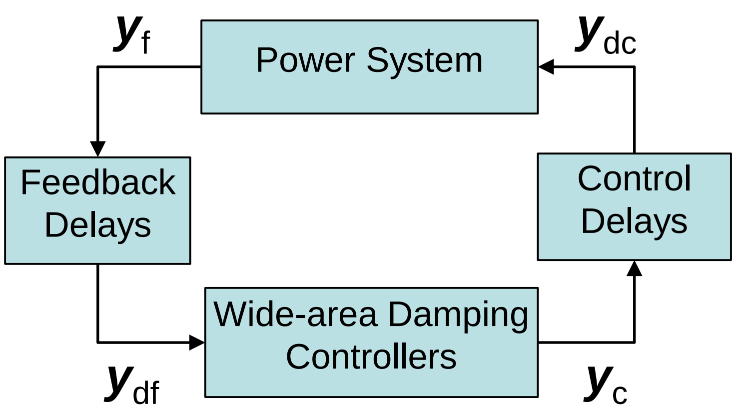


An illustrative diagram of the delayed cyber-physical power system (DCPPS)

- **Time delay** in the range of tens to hundreds of millisecond emerges in transmission and process of wide-area measurements.
- With the consideration of time delay effects, **the cyber-physical power system (CPPS) had been involved into a *Delayed CPPS* (*DCPPS*)**
- Time delays compromise the performance of wide-area control system and thus may jeopardize the stability of DCPPS.
- Its impact on small signal stability of DCPPS therefore should be intensively studied.

Formulation of SSSA of DCPPS

Structure of the DCPPS:



The dynamics of the DCPPS can be represented by the following linearized delayed differential equations (DDEs):

$$
\begin{cases}
\Delta\dot{\boldsymbol{x}}(t) = \tilde{A}_0 \Delta\boldsymbol{x}(t) + \sum_{i=1}^{m} \tilde{A}_i \Delta\boldsymbol{x}(t-\tau_i), \ t \geq 0 \\
\Delta\boldsymbol{x}(t) = \Delta\boldsymbol{x}_0 \triangleq \varphi, \ t \in [-\tau_{\max}, 0]
\end{cases}
$$

where τ_i (i=1, …, m) are delay constants.

Key problems related to small signal stability analysis of DCPPS:

1. Impacts of time delay on system's small signal stability
2. The maximum delay that the system can tolerate and maintain stable (i.e., delay margin)
3. Wide-area damping controller design by taking time delay impacts into consideration

Existing methods for SSSA of DCPPS

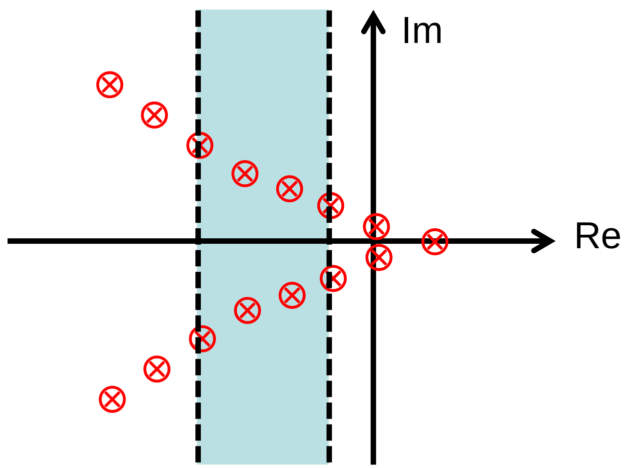
- The most popular methods to deal with DCPPS are: 1) delay-dependent stability criteria; 2) Padé approximation.
- Delay-dependent stability criteria
  - They are essentially Lyapunov functions and *sufficient conditions* for asymptotic stability. It is well understood that they are inherently conservative.
  - Its accuracy is further compromised since model reduction is always accompanied to reduce the cumbersome computational burden in analyzing large DCPPS.
  - Seldom of them can deal with DCPPS with *multiple* time delays.
- Padé approximation
  - Padé rational polynomial has been widely used in approximating an exponential delay term. Accordingly, wide-area damping controllers can be designed while time delay effects are considered.
  - The estimation accuracy for a *single* Padé approximation is analytic. However, it is not the case when multiple exponential delay terms are approximated by Padé rational polynomials.

2. Semi-group Discretization-based Eigen-analysis Framework

The characteristic equation of DCPPS is:

$$\left(\tilde{A}_0 + \sum_{i=1}^{m} \tilde{A}_i e^{-\lambda \tau_i}\right) \boldsymbol{v} = \lambda \boldsymbol{v}$$

where λ and $\boldsymbol{v}$ are the eigenvalue and the corresponding right-eigenvector.



Two conclusions on spectrum of DCPPS:

1. There are only a finite number of eigenvalues in any vertical strip of the complex plane.
2. The number of eigenvalues lying in the right-half complex plane are at most in a finite number.

Advantages:

1. Eigen-analysis is an elementary method for small signal stability analysis of power system.
2. It is capable of providing some useful metrics / indicators for stability analysis and control synthesis, such as damping ratio, oscillation frequency, participating factor, etc.

Challenges:

1. Since the exponential delay terms are involved, the characteristic equation of DCPPS is transcendental.
2. The equation has an infinite number of solutions (eigenvalues), which is basically unsolvable by traditional eigenvalue methods.

Solution:

To compute a set of rightmost eigenvalues of DCPPS by using the semi-group discretization-based eigen-analysis methods

Semi-group Dicretization-based Eigen-analysis Framework

Delayed differential equations of DCPPS

$$\Delta\dot{\boldsymbol{x}}(t) = \tilde{A}_0\Delta\boldsymbol{x}(t) + \sum_{i=1}^{m}\tilde{A}_i\Delta\boldsymbol{x}(t-\tau_i)$$

Semi-group operators

1. Solution operator $\mathcal{T}(h)$
2. Infinitesimal generator $\mathcal{A}$

Abstract equations (**ordinary functional equations**) in ***X***

$$\Delta\boldsymbol{x}_h = \mathcal{T}(h)\varphi$$

$$\Delta\dot{\boldsymbol{x}}_h = \mathcal{A}\Delta\boldsymbol{x}_h$$

(1) Spectral mapping

$$\lambda \in \sigma(\mathcal{A})$$

$$\lambda = \frac{\ln\mu}{h}, \mu \in \sigma(\mathcal{T}(h)) \setminus \{0\}$$

(2) Spectral discretization

$$(\Delta\boldsymbol{x}_h)^N = \mathcal{T}_N(\varphi)^N$$

$$(\Delta\dot{\boldsymbol{x}}_h)^N = \mathcal{A}_N(\Delta\boldsymbol{x}_h)^N$$

Linear multi-step
Implicit Runge-Kutta
Pseudo-spectral differencing

(3) Spectral estimation (sparse eigenvalue computation)

$$\mathrm{U}\hat{\lambda} = \sigma(\mathcal{A}_N), \mathrm{U}\hat{\mu} = \sigma(\mathcal{T}_N)$$

(4) Spectral correction

(Newton iteration)

$$\hat{\lambda} \to \lambda,\ \hat{\mu} \to \mu$$

** In mathematics, a **semigroup** is an algebraic structure consisting of a set together with an associative binary operation.

3. Infinitesimal generator discretization (IGD)-Based Methods for Large DCPPS

1. **Theoretical foundation**: eigenvalues λ of the DCPPS equal to those of the infinitesimal generator $\mathcal{A}$

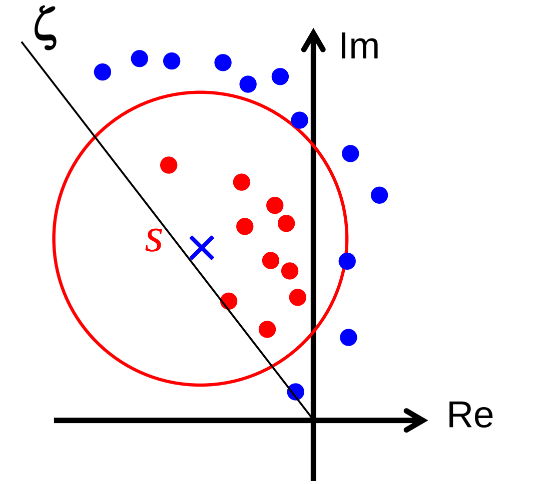
$$(1)\ \Delta\dot{x}_h = \mathcal{A}\Delta x_h; \quad (2)\left(\Delta\dot{x}_h\right)^N = \mathcal{A}_N\left(\Delta x_h\right)^N; \quad (3)\ \hat{\lambda} = \sigma(\mathcal{A}_N)$$

2. **Core Techniques**

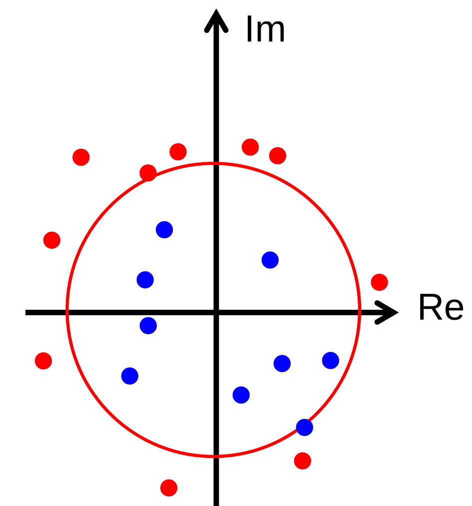
- **Shift-and-invert transformation**: eigenvalues λ of the DCPPS around a shift point s are transformed into the eigenvalues λ' of $(\mathcal{A}_N)^{-1}$ with the largest moduli.

$$\lambda' = \frac{1}{\lambda - s}$$

- **Kronecker product reformulation**: It lays the basis of utilizing the inherent sparsity in the augmented system state matrices.
- **Sparse eigenvalues computation**: the Arnoldi algorithm is utilized to efficiently compute the eigenvalues of $(\mathcal{A}_N)^{-1}$ with the largest moduli.



(a) Spectrum of DCPPS (or $\mathcal{A}$)



(b) Spectrum of $(\mathcal{A})^{-1}$

Principle of shift-invert transform

Idea of Kronecker product reformulation

After the shift-invert transformation, the discretized approximant matrix to the infinitesimal generator $\mathcal{A}$ becomes:

$$(\mathcal{A}'_N)^{-1} = (\Sigma'_N)^{-1}\Pi_N$$

$$\boldsymbol{R}'_0 = \tilde{A}_0 + \sum_{i=1}^{m} \tilde{A}_i$$

where

$$(\Sigma'_N)^{-1} = \begin{bmatrix} (\boldsymbol{R}'_0)^{-1} & -(\boldsymbol{R}'_0)^{-1}\boldsymbol{R}'_1 & \cdots & -(\boldsymbol{R}'_0)^{-1}\boldsymbol{R}'_N \\ & \boldsymbol{I}_n & & \\ & & \ddots & \\ & & & \boldsymbol{I}_n \end{bmatrix}$$

$$\boldsymbol{R}'_j = \tilde{A}_0 + \sum_{i=1}^{m} \tilde{A}_i T_j\left(-2\frac{\tau_i}{\tau_{\max}}+1\right),$$

$$j = 1, \cdots, N$$

To lay the basis of utilizing the inherent sparsity of the augmented system state matrices, the first block row is reformulated as **sums of Kronecker products between the constant vectors and system state matrices** $\tilde{A}_i$ **(*i*=0, 1, …, m)**.

$$\boldsymbol{\Gamma} = \begin{bmatrix} (\boldsymbol{R}'_0)^{-1} & -(\boldsymbol{R}'_0)^{-1}\boldsymbol{R}'_1 & \cdots & -(\boldsymbol{R}'_0)^{-1}\boldsymbol{R}'_N \end{bmatrix}$$

$$= \mathbf{e}_1^{\mathrm{T}} \otimes \boldsymbol{I}_n + \sum_{i=0}^{m} \mathbf{l}_i^{\mathrm{T}} \otimes \tilde{A}_i$$

where $\otimes$ denotes **the Kronecker product**.

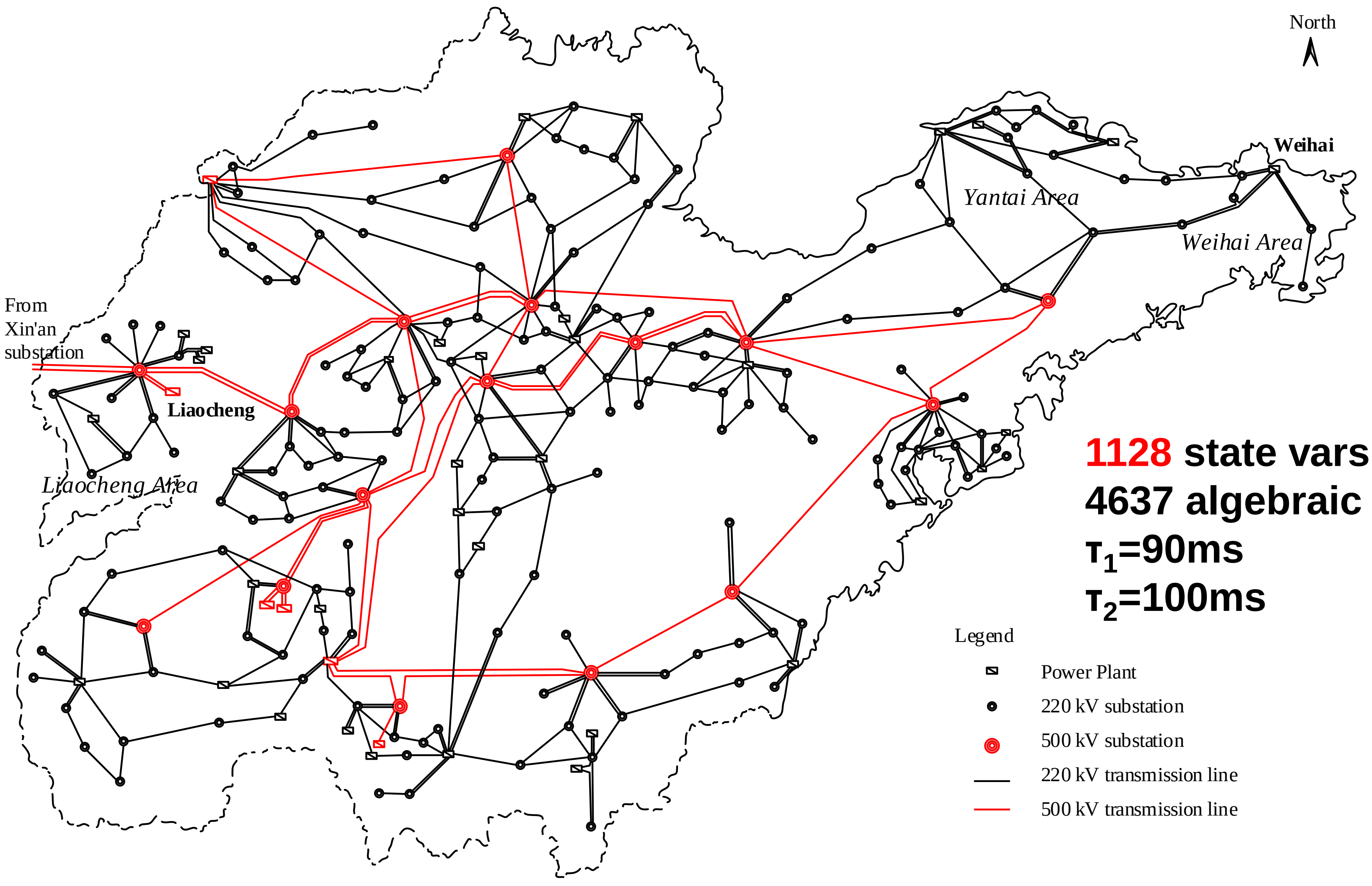
Compute λ' from $(\mathcal{A}_N)^{-1}$ by IRA algorithm

Let the kth Krylov vector be $\boldsymbol{q}_k$, the $(k+1)$th vector $\boldsymbol{q}_{k+1} = (\mathcal{A}'_N)^{-1}\boldsymbol{q}_k$ can be efficiently implemented as follows:

1. Compute $\boldsymbol{r} = \Pi_N \boldsymbol{q}_k \in \mathbb{R}^{(N+1)n\times 1}$;
2. Compress $\boldsymbol{r}$ into a matrix $\boldsymbol{V} = [\boldsymbol{v}_0, \ldots, \boldsymbol{v}_N] \in \mathbb{R}^{n\times(N+1)}$, $\boldsymbol{v}_j \in \mathbb{R}^{n\times 1}$, $j=0, \ldots, N+1$;
3. $\boldsymbol{z} = \boldsymbol{v}_0$;
4. for $j = 1, \ldots, N+1$
5. $\boldsymbol{p} = \tilde{A}_0 \boldsymbol{v}_j$
6. $\boldsymbol{w} = \sum_{i=1}^{m} \tilde{A}_i T_j\left(-2\tau_i/\tau_{\max}+1\right)\boldsymbol{v}_j$
7. $\boldsymbol{z} = \boldsymbol{z} - \boldsymbol{p} - \boldsymbol{w}$;
8. end
9. Compute $\boldsymbol{q}_{k+1}(1:n) = (\boldsymbol{R}'_0)^{-1}\boldsymbol{z}$;
10. $\boldsymbol{q}_{k+1}((n+1):(N+1)n) = \boldsymbol{r}((n+1):(N+1)n)$.

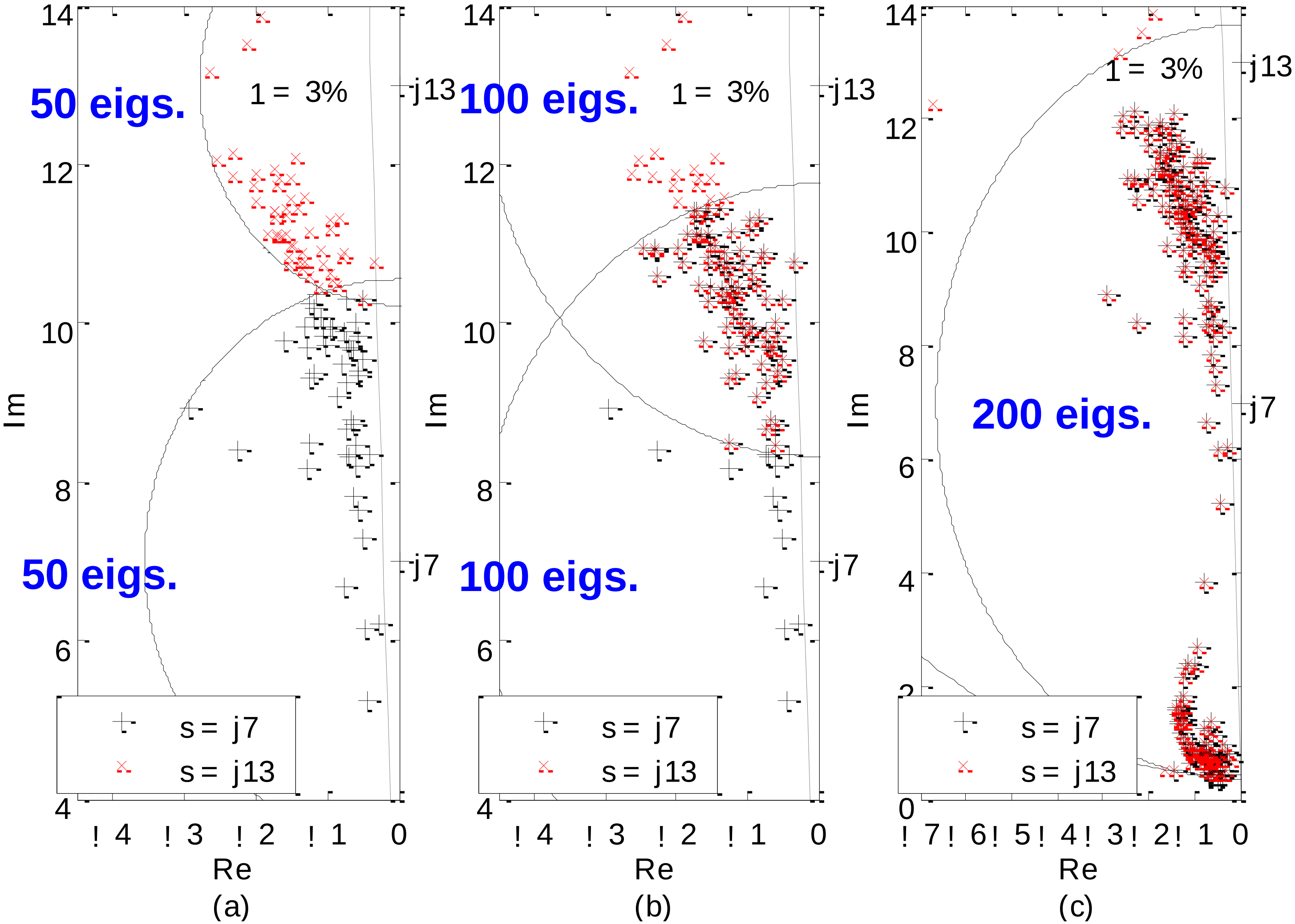
1. The two matrix-vector products can be efficiently implemented by using the inherent sparsity in system state matrices $\tilde{A}_i$ ($i=0,1, \ldots, m$).
2. The computational burden of the EIGD algorithm nearly amounts to $N+1$ times of traditional eigenvalue computation.
3. Estimate to eigenvalue of DCPPS is restored by $\lambda = s + 1/\lambda'$.

Test results of Shandong power grid (n=1128)



The backbone of Shandong power grid

1128 state vars.
4637 algebraic vars.
τ_1=90ms
τ_2=100ms



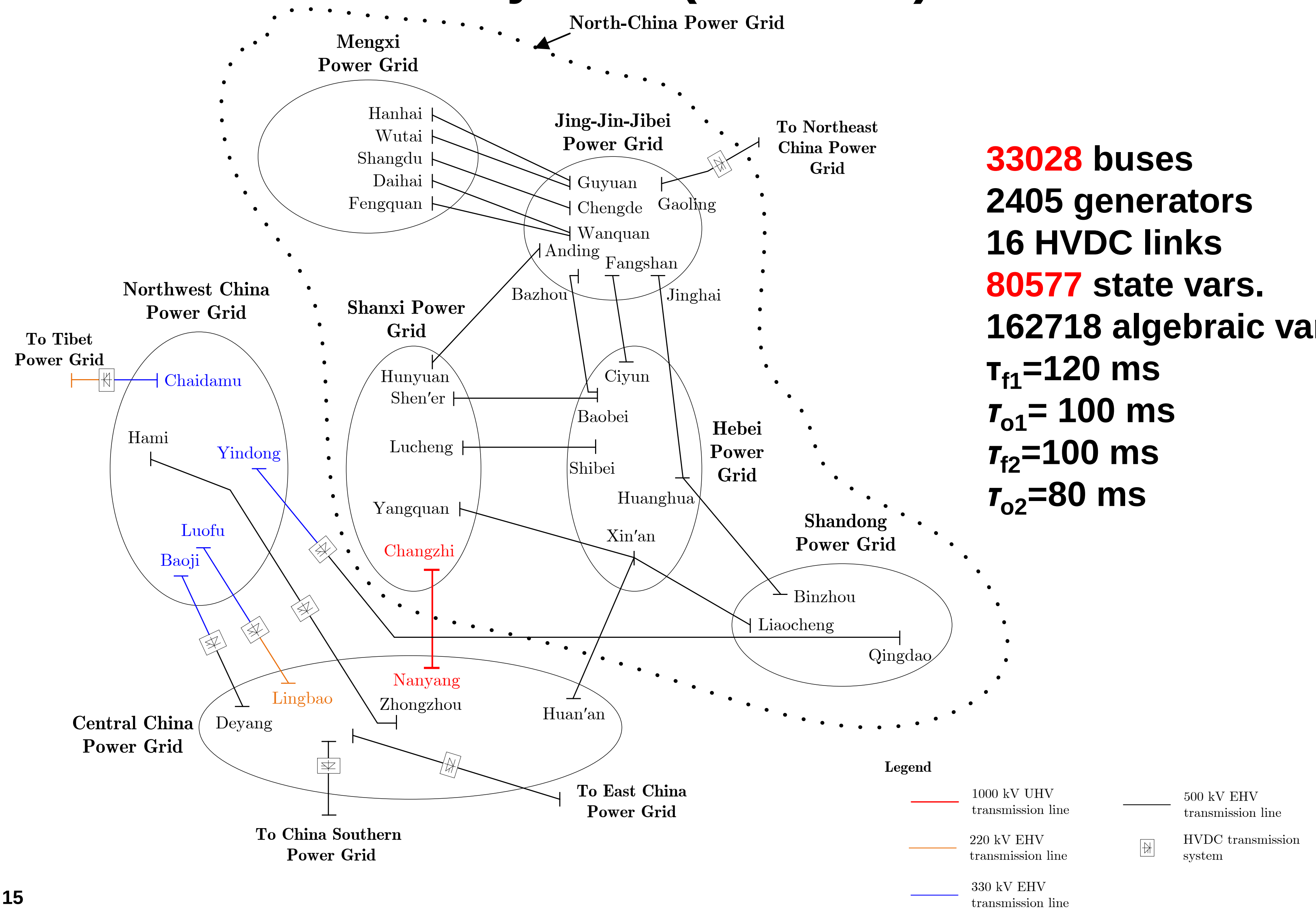
For shift points 7j and 13j, *r* eigenvalues of $\mathcal{A}_N$.

(a) *r*=50; (b) *r*=100; (c) *r*=200.

Computational Efficiency Analysis
[# of Arnoldi Iterations / CPU time /s]

Method	N	dim($\mathcal{A}_N$)	Shift s	r eigenvalues computed		
				50	100	200
EIGD (n=1128, m=2)	20	23688	j7	7 / 15.37	2 / 24.21	6 / 187.63
	20	23688	j10	3 / 7.93	2 / 21.49	9 / 264.57
	20	23688	j13	4 / 10.55	2 / 21.47	11 / 317.5
	40	46248	j7	7 / 30.34	2 / 46.79	6 / 353.95
	40	46248	j10	2 / 11.75	2 / 41.36	9 / 501.91
	40	46248	j13	4 / 21.20	2 / 41.26	11 / 602.3

Test results of North China-Central China interconnected system (n=33028)



33028 buses
2405 generators
16 HVDC links
80577 state vars.
162718 algebraic vars.
τ_{f1}=120 ms
τ_{o1}= 100 ms
τ_{f2}=100 ms
τ_{o2}=80 ms

Computational Efficiency Analysis

[# of Arnoldi Iterations / CPU time /s]

Method	N	dim($\mathcal{A}_N$)	Shift s	r eigenvalues computed	
				10	20
EIGD (n=80577, m=4)	20	1692117	j7	7 / 139.44	6 / 347.59
	20	1692117	j10	4 / 87.38	3 / 153.76
	20	1692117	j13	6 / 140.14	6 / 351.28

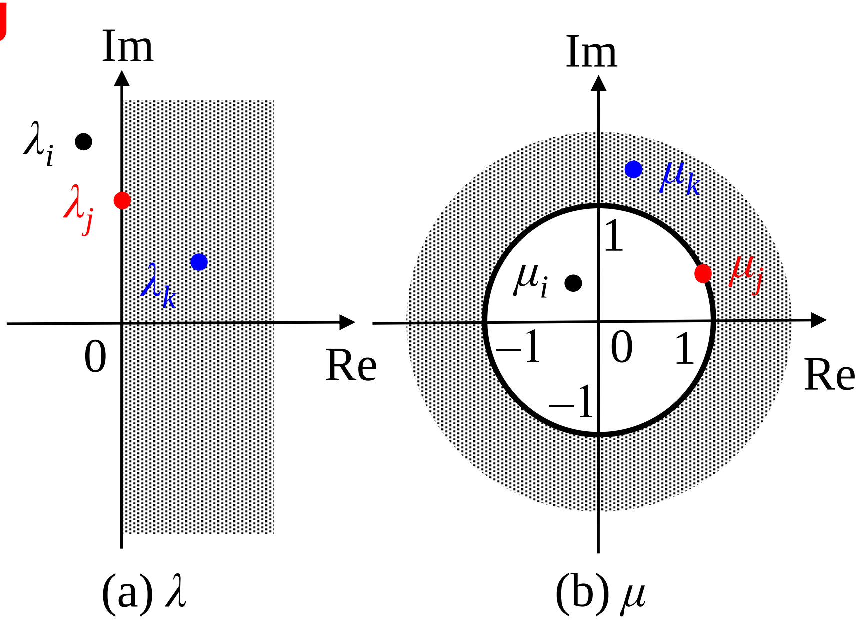
4. Solution operator discretization (SOD)-Based Methods for Large DCPPS

1. **Theoretical foundation**: the eigenvalues λ of the DCPPS relate to those of the solution operator $\mathcal{T}(h)$ as a logarithm function.

$$(1)\,\Delta \boldsymbol{x}_h = \mathcal{T}(h)\varphi;\quad (2)\left(\Delta \boldsymbol{x}_h\right)^N = \boldsymbol{T}_N \varphi^N;\quad (3)\,\hat{\lambda} = \frac{1}{h}\ln \hat{\mu},\ \hat{\mu} = e^{h\hat{\lambda}}$$

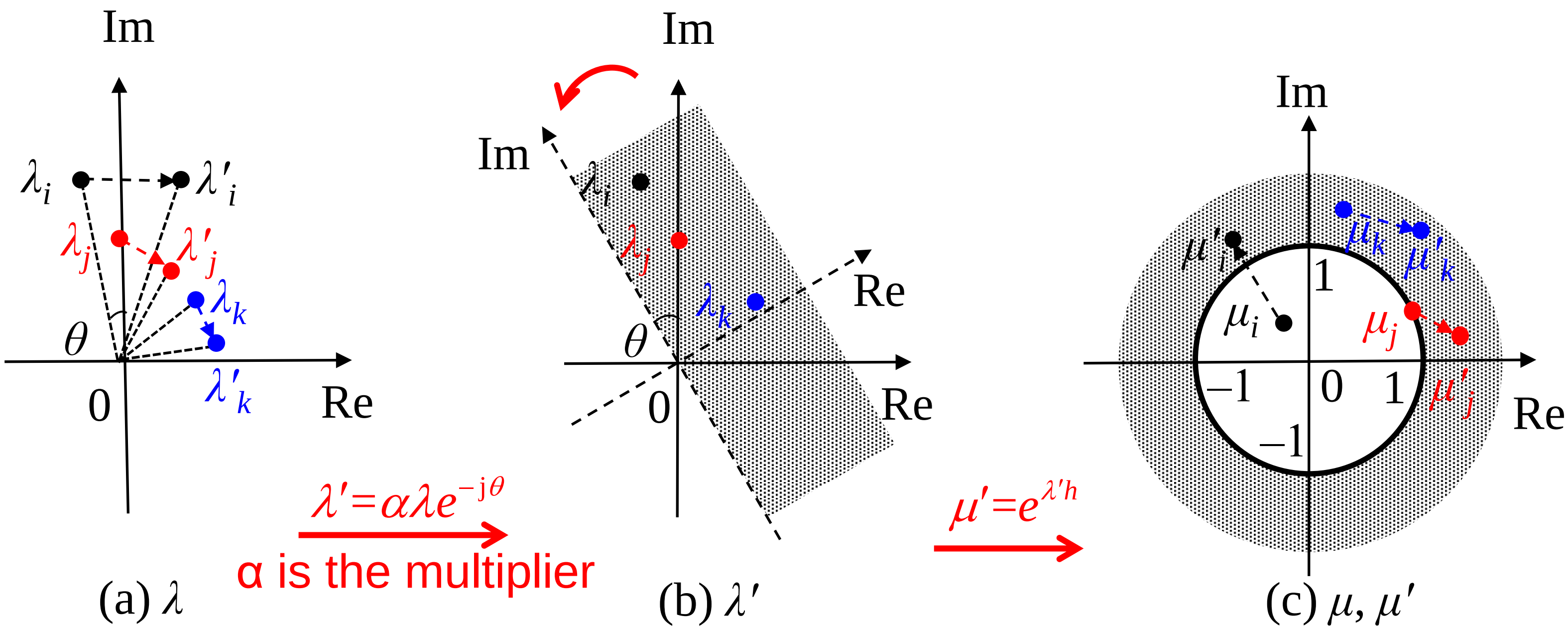
2. **Unique properties of the spectral mapping**
   - The eigenvalues λ of DCPPS located on the left-half of the *s*-plane are transformed into those of $\mathcal{T}(h)$ situated inside of unit circle on the *z*-plane.
   - For a fixed *h*, the real part of λ, i.e., Re(λ), monotonically increases with $|\mu|$.

3. **Application of the spectral mapping properties to analyze DCPPS**
   - **Reliable stability determination**: If the largest modulus $|\mu_1|<1$, the DCPPS is asymptotically stable. If modulus $|\mu_1|>1$, the system is unstable.
   - **Critical eigenvalue computation**: The rightmost eigenvalues λ of DCPPS with the largest real parts can be recovered from μ with the largest moduli.



4. Core technique ------ Rotation-and-multiplication preconditioning

Eigenvalues of the DCPPS **with damping ratios less than a given threshold** are transformed into those of $\mathcal{T}(h)$ with the largest moduli.



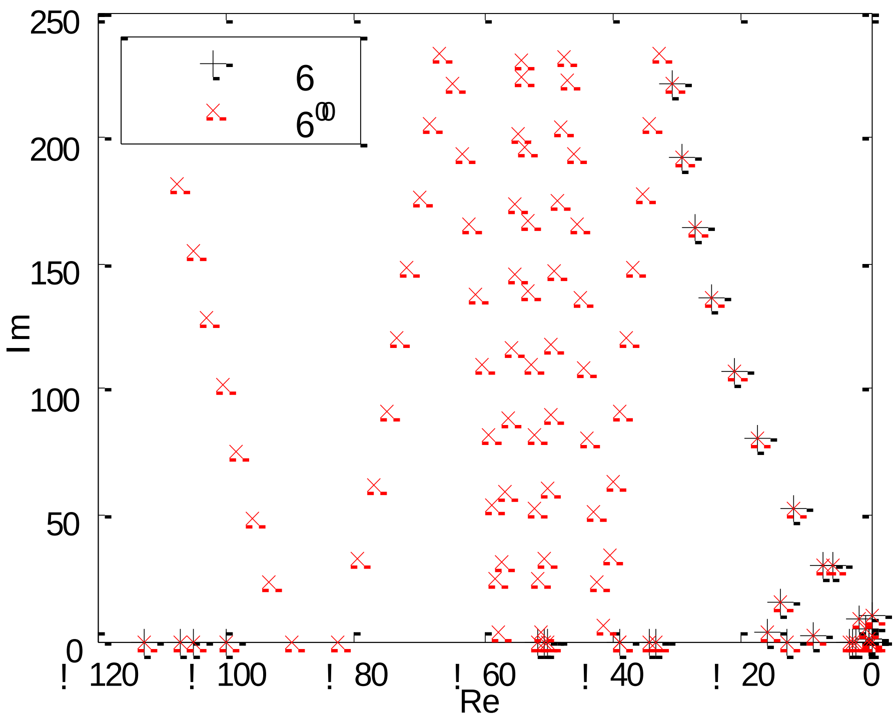
The preconditioning improves the distribution of eigenvalues μ to speed up the convergence rate of sparse eigenvalue computation method.

5. Algorithms

- SOD-PS (Trans. PWRS 2017, 2017)
- SOD-LMS, SOD-IRK (submitted to Trans. PWRS, 2017)

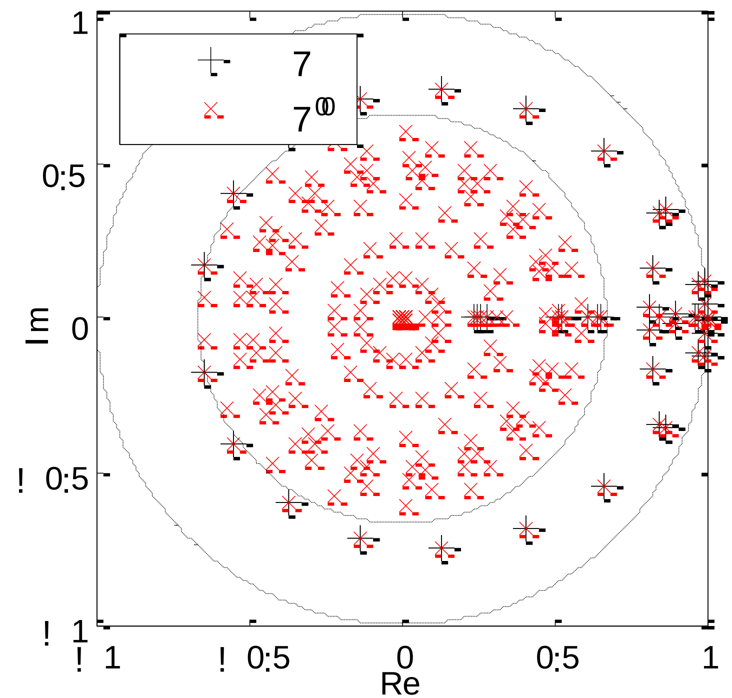
Results of the 16-machine 68-bus test system

Eigenvalues λ of DCPPS



(a) *λ″, λ*

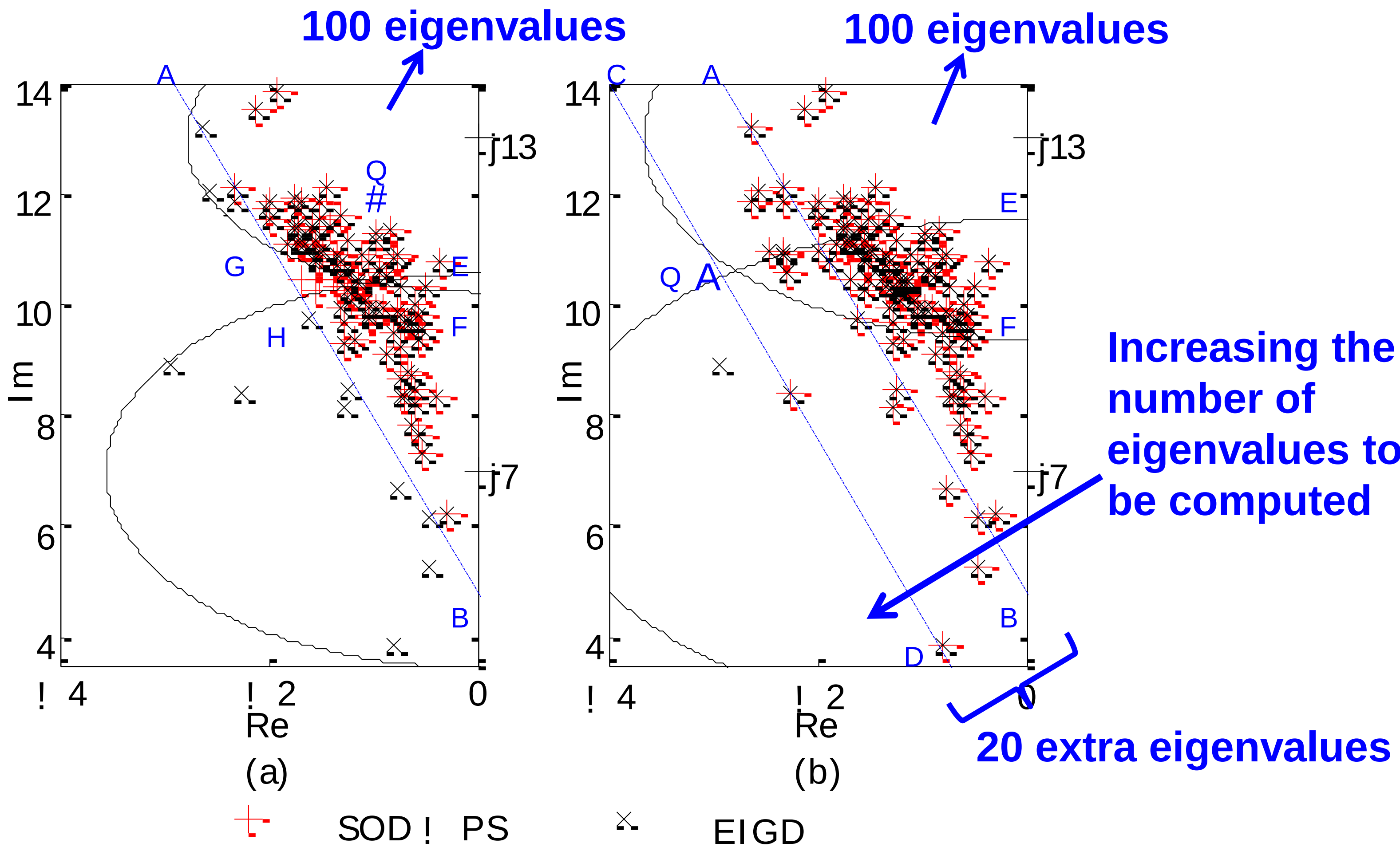
Eigenvalues μ of the solution operator



(b) *μ″, μ*

1. The accurate eigenvalues λ of DCPPS are located on the right of the spurious eigenvalues.
2. For the solution operator, the accurate eigenvalues μ are with larger moduli and located on the outside of the inner circle.

Results of the Shandong power grid (n=1128)



r = 100 and 120 Eigenvalues of DCPPS computed by SOD-PS and EIGD. For SOD-PS, θ=17.46° and α=2; For EIGD, shifts s=j7 and j13.

Results of North China-Central China interconnected system (n=33028)

Computational Efficiency Analysis

[# of Arnoldi Iterations / CPU time /s]

Methods	***k***	***s***	***p***	***q***	***N***	***h***	**Dim**	**N_{IRA}/Time(s)**
SOD-LMS	3	-	-	-	20	0.006	1,047,501	**177 / 7358.6**
SOD-IRK	-	2	-	-	20	0.006	1,611,540	**164 / 19543**
SOD-PS	-	-	3	3	20	0.006	1,128,078	**155 / 45009**

SOD-LMS method is more efficient, while SOD-IRK and SOD-PS are more accurate.

5. Conclusions

- The proposed semi-group discretization-based eigen-analysis framework for large DCPPS contains four main parts:
  - **Spectral mapping**: the eigenvalues λ of DCPPS are transformed into those of the infinitesimal generator $\mathcal{A}$ and solution operator $\mathcal{T}(h)$.
  - **Spectral discretization**: A number of numerical schemes, such as linear multi-step, implicit Runge-Kutta, pseudo-spectal differencing, can be adopted to implement $\mathcal{A} \rightarrow \mathcal{A}_N$, $\mathcal{T}(h) \rightarrow \boldsymbol{T}_N$
  - **Spectral estimation:** sparse eigenvalue computation methods (e.g. IRA) can be used to compute a set of critical eigenvalues of DCPS, while 1) shift-invert transform, rotation-and-multiplication preconditioning are necessary to enhance the convergence speed; 2) the inherent sparsity of the augmented system state matrices should be exploited to reduce the computational burden.
  - **Spectral correction**: The newton iteration is used to implement $\hat{\lambda} \rightarrow \lambda$.
- The proposed SOD and IGD methods can efficiently compute a set of critical eigenvalues of real-life DCPPSs.

Further references

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- **Hua Ye**, Qianying Mou, Yutian Liu. Enabling highly efficient spectral discretization-based eigen-analysis methods by Kronecker product. ***IEEE Transactions on Power Systems***, 2017, pp. 1-3, early access.

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- **Hua Ye**, Yutian Liu, Peng Zhang. Efficient eigen-analysis for large delayed cyber-physical power system using explicit infinitesimal generator discretization. ***IEEE Transactions on Power Systems***, 2016, 31(3): 2361-2370.

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Thanks!